Effects of disorder on quantum fluctuations and superfluid density of a Bose-Einstein condensate in a two-dimensional optical lattice

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We investigate a Bose-Einstein condensate trapped in a 2D optical lattice in the presence of weak disorder within the framework of the Bogoliubov theory. In particular, we analyze the combined effects of disorder and an optical lattice on quantum fluctuations and superfluid density of the BEC system. Accordingly, the analytical expressions of the ground state energy and quantum depletion of the system are obtained. Our results show that the lattice still induces a characteristic 3D to 1D crossover in the behavior of quantum fluctuations, despite the presence of weak disorder. Furthermore, we use the linear response theory to calculate the normal fluid density of the condensate induced by disorder. Our results in the 3D regime show that the combined presence of disorder and lattice induce a normal fluid density that asymptotically approaches 4/3 of the corresponding condensate depletion. Conditions for possible experimental realization of our scenario are also proposed.

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I. INTRODUCTION

Ultracold atoms in an optical lattice have opened a new theoretical and experimental window for investigating fundamental problems related to condensed matter physics in a very versatile manner [1, 2, 3]. Bose-Einstein condensates (BEC) trapped in an optical lattice allow for optimal control of the system's parameters: the depth of an optical lattice can be arbitrarily modified by changing the intensities of laser beams [4], whereas the interatomic interactions can be controlled via the technology of Feshbach resonances [5]. Having at hand the possibility to shape the optical lattices and realize quasi-1D, quasi-2D and 3D BEC systems almost at will, therefore, an important direction of investigation consists in studying the properties of a BEC system along the dimensional crossover.

Along this line, much work have been done [1, 2]. In particular, Orso *et al.*[6] theoretically showed that a 2D lattice can induce a characteristic 3D to 1D crossover in the behavior of quantum fluctuations. Interest in this dimensional crossover needs to be renewed, however, following the remarkable observation that disorder in quantum systems can have dramatic effects on the quantum fluctuations and superfluid density of a BEC [7, 8, 9, 10, 11, 12, 13, 14, 15]. For example, it has been observed that even a tiny amount of disorder in the confining fields leads to a fractioning of quasi-1D condensates in waveguide structures on atom chips [16]. Hence, a natural question that immediately arises is how the disorder affects the 2D-lattice-induced dimensional crossover predicted in Ref. [6].

In this paper, we launch a systematic investigation on a BEC in the combined presence of 2D optical lattice and weak external randomness. The present work is divided into two parts. In the first part, we shed new light on the interplay between disorder and interatomic interaction along the 2D-lattice-induced dimensional crossover. Accordingly, the analytical expressions of the ground state energy and quantum fluctuations are derived for a BEC in a tight 2D optical lattice in the presence of disorder using the Bogoliubov theory. Our results show that for a fixed atomic density, the BEC at small values of the lattice depth is in an anisotropic 3D regime. When the lattice depth increases, however, the BEC undergoes a dimensional crossover from an anisotropic 3D regime to a 1D regime. The effects of disorder on such dimensional crossover is analyzed in detail. Our results for the ground state energy in the asymptotic 1D regime without disorder is in agreement with the exact Lieb-Linger solution expanded in the weak coupling regime [17]. We argue, therefore, that our result generalizes the exact ground state energy of Lieb-Linger model to that in the presence of weak disorder. All of our results in case of vanishing disorder are consistent with those obtained in Ref. [6].

In the second part of this work, we calculate the normal fluid density of the condensate induced by disorder using the linear response theory. Accordingly, the transverse current-current response functions are calculated. These results in case of vanishing lattice depth recover corresponding ones in Ref. [18]. In particular, our results in the 3D regime show that the combined presence of disorder and lattice induce a normal fluid density that asymptotically approaches 4/3 of the corresponding condensate depletion [18].

The above scenario for disorder can be realized in cold

atomic systems using several methods in a controlled way. They include applying optical potentials created by laser speckles or multi-chromatic lattices [19, 20], introducing impurity atoms in the sample [21] and manipulating the collision between atoms [22]. The controllability of disorder in a BEC system makes the studies of disorder-induced effects extremely fascinating [23, 24, 25].

The outline of this paper is as follows. In Sec. II, we introduce the Hamiltonian for a BEC trapped in a 2D optical lattice in the presence of weak disorder and then analyze this model using the Bogoliubov approximation. In Sec. III, the analytical expressions for the ground state energy and quantum depletion of such BEC system are obtained. Especially, we focus on the analysis of the combined effects of disorder and optical lattice on quantum fluctuations of this BEC system . Sec. IV presents calculations of reduction of the superfluid density due to disorder trapped in an optical lattice. Finally in Sec. V, we summarize our results and propose experimental conditions for realizing our scenario.

II. HAMILTONIAN FOR A BEC IN BOTH PRESENCE OF A 2D OPTICAL LATTICE AND WEAK DISORDER

The N-body Hamiltonian describing the Bose system has the form

$$H-\mu N=\int d\mathbf{r}\hat{\Psi}^{\dagger}(\mathbf{r})\Big[-\frac{\hbar^2\nabla^2}{2m}-\mu+V_{opt}(\mathbf{r})+V_{ran}(\mathbf{r}) + \frac{g}{2}\hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}(\mathbf{r})\Big]\hat{\Psi}(\mathbf{r}), \quad (1)$$

where $\hat{\Psi}(\mathbf{r})$ is the field operator for bosons with mass m, μ is the chemical potential, $N=\int d\mathbf{r}\hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}(\mathbf{r})$ is the number operator, and $g=4\pi\hbar^2 a/m$ is the coupling constant with a being the 3D scattering length in the free space. In the Hamiltonian (1), $V_{ran}(\mathbf{r})$ and $V_{opt}(\mathbf{r})$ respectively represent the external random potential and the 2D optical lattice.

The 2D optical potential $V_{opt}(\mathbf{r})$ in the Hamiltonian (1) is given by

$$V_{opt}(\mathbf{r})=sE_R\left[\sin^2(q_B x)+\sin^2(q_B y)\right], \quad (2)$$

where s is a dimensionless factor labeled by the intensity of the laser beam and $E_R=\hbar^2 q_B^2/2m$ is the recoil energy with $\hbar q_B$ being the Bragg momentum. The lattice period is fixed by π/d. Atoms are unconfined in the z-direction.

Disorder in the Hamiltonian (1) is produced by the random potential associated with quenched impurities [26]

$$V_{ran}(\mathbf{r})=\sum_{i=1}^{N_{imp}} v\left(|\mathbf{r}-\mathbf{r_i}|\right), \quad (3)$$

with $v(\mathbf{r})$ describing the two-body interaction between bosons and impurities, $\mathbf{r}_i$ being the randomly distributed positions of impurities and N_{imp} counting the number of $\mathbf{r}_i$. To obtain the concrete form of the pair potential $v(\mathbf{r})$, we need to investigate the scattering problem between a particle and a quenched impurity whose mass is taken to be infinite in the presence of 2D optical lattice. Here, we restrict ourself to the conditions of a dilute BEC system in the presence of a very small concentration of disorder. Thereby, the potential $v(\mathbf{r})$ can be expressed by a pseudo-potential $v(\mathbf{r})=\tilde{g}_{imp}\delta(\mathbf{r})$. Here, the $\tilde{g}_{imp}$ is the effective coupling constant that reads $\tilde{g}_{imp}=2\pi\hbar^2\tilde{b}/m$, where $\tilde{b}$ represents the effective scattering length accounting for the presence of a 2D optical lattice [27].

In what follows, we will assume that the optical lattice is strong enough to create many separated wells that give rise to an array of condensates. Meanwhile, because of the quantum tunneling, the overlap between the wave functions of two consecutive wells are still sufficient to ensure full coherence even in the presence of disorder. In such case, one is allowed to use the Bogoliubov theory to study both equilibrium and dynamic behavior of the system at zero temperature. In addition, we also suppose that the chemical potential μ is small compared to the inter-band gap. By this assumption, we restrict ourselves to the lowest band, where the physics is governed by the ratio between the chemical potential μ and the bandwidth $8t$, where t is the tunneling rate between neighboring wells. Generally speaking, for $\mu \ll 8t$, the system retains an anisotropic 3D behavior, whereas for $\mu \gg 8t$, the system undergoes a dimensional crossover to a 1D regime.

In the tight-binding approximation [28], the lowest Bloch band of the BEC system can be written in terms of Wannier functions as $\phi_{k_x}(x)\phi_{k_y}(y)$, where $\phi_{k_x}(x)=\sum_l e^{ilkx}w(x-ld)$ and $w(x)=\exp[-u^2/2\sigma^2]/\pi^{1/4}\sigma^{1/2}$ with $d/\sigma\simeq\pi s^{1/4}\exp(-1/4\sqrt{s})$. Expanding the field operators by the expression $\hat{\Psi}(\mathbf{x})=\sum_{\mathbf{k}}\hat{a}_{\mathbf{k}}e^{-ik_z z}\phi_{k_x}(x)\,\phi_{k_y}(y)$, the Hamiltonian (1) takes the form

$$\begin{aligned} H^{'} &= H-\mu N \\ &= \sum_{\mathbf{k}}\left(\epsilon^0_{\mathbf{k}}-\mu\right)\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}+\frac{\tilde{g}}{2V}\sum_{\mathbf{k},\mathbf{q},\mathbf{k'}}\hat{a}^{\dagger}_{\mathbf{k}+\mathbf{q}}\hat{a}^{\dagger}_{\mathbf{k'}-\mathbf{q}}\hat{a}_{\mathbf{k'}}\hat{a}_{\mathbf{k}} \\ &+ \sum_{\mathbf{k},\mathbf{k'}}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k'}}V_{\mathbf{k}-\mathbf{k'}}, \end{aligned} \quad (4)$$

where $\epsilon^0_{\mathbf{k}}=\hbar^2k_z^2/2m+2t[2-\cos(k_x d)-\cos(k_y d)]$ is the energy dispersion of the noninteracting model, V is the volume and $\tilde{g}=4\pi\hbar^2\tilde{a}/m$ is an effective coupling constant with $\tilde{a}=C^2a$ and $C=d\int_{-d/2}^{d/2}w^4(x)dx\simeq d/\sqrt{2\pi}\sigma$. In Eq. (4), the $V_{\mathbf{k}}$ is the Fourier transform of $V_{ran}(\mathbf{r})$

$$V_{\mathbf{k}}=\frac{1}{V}\int e^{i\mathbf{k}\cdot\mathbf{r}}V_{ran}(\mathbf{r})d\mathbf{r}. \quad (5)$$

Here we choose that the randomness is uniformly distributed with density $n_{imp}=N_{imp}/V$ and Gaussian correlated [26]. Thereby, the two basic statistical properties of disorder are the average value $\langle V_0\rangle=\tilde{g}_{imp}n_{imp}$ and

the correlation function $\langle V_{\mathbf{k}} V_{-\mathbf{k}}\rangle = \tilde{g}_{imp}^2 n_{imp}/V$. Here, the notation $\langle ... \rangle$ stands for the ensemble average over all disorder configurations [7].

In the following, we focus on the situation where the number of atoms in each tube is sufficiently large and refer to n_0 as the condensate density. Under this assumption [29], we can neglect the Mott insulator phase transition which would occur only for extremely large values of the lattice depth. Throughout this paper, we consider the case when the strength of disorder is weak enough and Bogoliubov approximation still holds [26]. Hence, by proceeding in a standard fashion of Bogoliubov theory, the Hamiltonian (4) can be approximated as

$$
\begin{aligned}
H_{eff} - \mu N \; &= \; V\left(-\mu n_0 + n_0 \tilde{V}_0 + \frac{1}{2}\tilde{g} n_0^2\right) + \frac{1}{2}\sum_{\mathbf{k}\neq 0}\left(\varepsilon_{\mathbf{k}}^0 - \mu + 2\tilde{g} n_0\right)\left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}\right) \\
&+ \; \frac{1}{2}\tilde{g} n_0 \sum_{\mathbf{k}\neq 0}\left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}}\right) + \sqrt{n_0}\sum_{\mathbf{k}\neq 0}\left(\hat{a}_{\mathbf{k}}^\dagger V_{-\mathbf{k}} + \hat{a}_{\mathbf{k}} V_{\mathbf{k}}\right) + \frac{\tilde{g}}{V}\left[\sum_{\mathbf{k}\neq 0}\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}\right]^2 . \qquad (6)
\end{aligned}
$$

In the effective Hamiltonian (6), the only processes considered are the annihilation of a pair $\{\mathbf{k}, -\mathbf{k}\}$ into the condensate through the two-body interaction, the scattering of a single particle $\mathbf{k}$ into the condensate by disorder, and the corresponding inverse processes. When the condensate becomes depleted, the last term in Hamiltonian (6) is important [18, 30] and can be treated by making the replacement $\frac{1}{V}\left[\sum_{\mathbf{k}\neq 0}\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}\right]^2 \to n' \sum_{\mathbf{k}\neq 0}\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$, where n' is a parameter to be determined later. We emphasize here that the introduction of n' in Eq. (6) is equivalent to the introduction of an additional Lagrange multiplier as shown in the Refs. [3, 7, 31]. The effective Hamiltonian (6) is then diagonalized by the Bogoliubov transformation

$$
\begin{aligned}
\hat{a}_{\mathbf{k}} \; &= \; u_{\mathbf{k}} \hat{c}_{\mathbf{k}} - \upsilon_{\mathbf{k}} \hat{c}_{-\mathbf{k}}^\dagger - \frac{\sqrt{N_0}(u_{\mathbf{k}} - \upsilon_{\mathbf{k}})^2}{E_{\mathbf{k}}} V_k, \\
\hat{a}_{\mathbf{k}}^\dagger \; &= \; u_{\mathbf{k}} \hat{c}_{\mathbf{k}}^\dagger - \upsilon_{\mathbf{k}} \hat{c}_{-\mathbf{k}} - \frac{\sqrt{N_0}(u_{\mathbf{k}} - \upsilon_{\mathbf{k}})^2}{E_{\mathbf{k}}} V_{-k}, \qquad (7)
\end{aligned}
$$

with $\upsilon_{\mathbf{k}}^2 = u_{\mathbf{k}}^2 - 1 = \frac{1}{2}\left(\frac{\varepsilon_k^0 + n_0\tilde{g}}{E_{\mathbf{k}}} - 1\right)$, $E_{\mathbf{k}} = \sqrt{(\varepsilon_k^0 + \Delta)^2 + 2n_0\tilde{g}(\varepsilon_k^0 + \Delta)}$, and $\Delta = \tilde{g}\,(n_0 + n') - \mu$. Consequently, the diagonalized Hamiltonian reads:

$$
H_{eff} - \mu N = -V\mu n_0 + E_g + \sum_{\mathbf{k}\neq 0} E_{\mathbf{k}} \hat{c}_{\mathbf{k}}^\dagger \hat{c}_{\mathbf{k}}, \qquad (8)
$$

with the ground state energy E_g reading

$$
\begin{aligned}
E_g \; &= \; \frac{1}{2} V \tilde{g} n_0^2 - \frac{1}{2}\sum_{k\neq 0}(\varepsilon_k^0 + n_0\tilde{g} - E_k) \\
&+ \; N\left[n_{imp}\tilde{g}_{imp} - \frac{n_{imp} g_{imp}^2}{V}\sum_{k\neq 0}\frac{\varepsilon_k^0}{E_k^2}\right]. \qquad (9)
\end{aligned}
$$

In conformity with general theorem [32], here we set $\Delta = \tilde{g}(n_0 + n') - \mu = 0$ to ensure a gapless quasiparticle spectrum [31]. This condition, together with $n = n_0 + 1/V\sum_{\mathbf{k}\neq 0}\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$, determines n' and n_0 as a function of n.

III. QUANTUM FLUCTUATIONS AND DIMENSIONAL CROSSOVER

A. Beyond mean field correction to the ground state energy

By replacing the sum in Eq. (9) with integrals in the continuum limit, we obtain

$$
\begin{aligned}
\frac{E_g}{V} \; &= \; \frac{1}{2}\tilde{g} n_0^2 - \frac{1}{4\pi}\frac{\tilde{g} n_0}{d^2}\sqrt{2m\tilde{g} n_0}\, f\left(\frac{2t}{\tilde{g} n_0}\right) \\
&+ \left[\frac{1}{2}\tilde{g} n_0^2\left(\kappa\frac{\tilde{b}}{\tilde{a}}\right) - \frac{\tilde{R}\tilde{g} n_0}{4\pi\hbar d^2}\sqrt{m n_0 \tilde{g}}\, Q\!\left(\frac{2t}{\tilde{g} n_0}\right)\right], \qquad (10)
\end{aligned}
$$

with $\kappa = n_{imp}/n_0$ and $\tilde{R} = \kappa\tilde{b}^2/\tilde{a}^2$. The last two terms in Eq. (10) are the beyond mean-field correction to the ground state energy due to disorder. The functions $f(x)$ and $Q(x)$ in Eq. (10) are respectively defined as

$$
f(x) = \frac{\pi}{2\sqrt{x}}\int_{-\pi}^{\pi}\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{{}_2F_1(\frac{1}{2}, \frac{3}{2}, 3, -\frac{2}{x\gamma(\mathbf{k})})}{\sqrt{\gamma(\mathbf{k})}}, \qquad (11)
$$

and

$$
Q(x) = \frac{\pi}{2}\int_{-\pi}^{\pi}\frac{dk_x}{2\pi}\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, \frac{x^2}{(1+x+x\sin^2(k_x/2))^2})}{\sqrt{x\sin^2(k_x/2) + 1}}. \qquad (12)
$$

In both Eqs. (11) and (12), the variable x stands for $2t/\tilde{g} n_0$ that appears in Eq. (10). In Eq. (11), $\gamma(\mathbf{k}) = 2 - \cos(k_x) - \cos(k_y)$. The function ${}_2F_1(a, b, c, d)$ in Eqs. (11) and (12) is the hypergeometric function [33] and the integration over the transverse quasimomenta is

restricted to the first Brillouin zone, i.e. $|k_x|, |k_y| \leq \pi$ in Eq. (11) and $|k_x| \leq \pi$ in Eq. (12).

Eq. (10) describes the ground state energy of a BEC in a combined presence of tight 2D periodic potential and weak disorder. Our model is characterized by three parameters: (i) the BEC parameter $n\tilde{a}^3$, (ii) the concentration of disorder $\kappa = n_{imp}/n_0$, and (iii) the ratio of effective scattering amplitudes $\tilde{b}/\tilde{a}$. The first one reflects the strength of interatomic interaction in the presence of periodic potential, whereas the other two are important parameters that describe the effect of disorder trapped in optical lattice.

B. Lattice-induced dimensional crossover in the presence of weak disorder

We now proceed to consider the asymptotic behavior of the ground state in the limit of $x \to 0$ and $x \gg 1$, respectively. In the limit of $x \gg 1$, corresponding to $8t \gg \mu$, the system retains an anisotropic 3D behavior; whereas for $x \to 0$, corresponding to $8t \ll \mu$, the system undergoes a dimensional crossover to a 1D regime.

In the 1D regime $x \to 0$, the $f(x)$ saturates to the value $4\sqrt{2}/3$ [6]. In this limit, we can neglect the Bloch dispersion and Eq. (8) approaches asymptotically to the ground state energy of a 1D Bose gas in the presence of weak disorder

$$\begin{aligned}\frac{E_g}{L} &= \frac{1}{2}g_{1D}n_{1D}^2 - \frac{2}{3\pi}\sqrt{m}\,(n_{1D}g_{1D})^{3/2} \\ &+ \Big[\frac{1}{2}g_{1D}n_{1D}^2\left(\kappa\frac{\tilde{b}}{\tilde{a}}\right) \\ &\quad -\frac{\tilde{R}\sqrt{m}}{4\pi\hbar}(n_{1D}g_{1D})^{3/2}Q\left(\frac{2t}{g_{1D}n_{1D}}\right)\Big], \end{aligned} \tag{13}$$

with linear density $n_{1D} = n_0 d^2$ and coupling constant $g_{1D} = \tilde{g}/d^2$. Here L is the length of the tube. In the case of vanishing disorder ($\kappa = 0$), Eq. (13) is in agreement with the exact Lieb-Linger solution of the 1D model expanded in the weak coupling regime $mg_{1D}/\hbar n_{1D} \ll 1$ [17]. We argue, therefore, that Eq. (13) with $\kappa \neq 0$ is the generalized Lieb-Linger solution of the 1D model expanded in the weak coupling regime in the presence of weak disorder.

In the opposite 3D regime $x \gg 1$, the functions (11) and (12) approach respectively the asymptotic law $f(x) \simeq 1.43/\sqrt{x} - 16\sqrt{2}/(15\pi x)$ and $Q(x) \simeq -1/\sqrt{\pi x}$. Hence in the limit $x \gg 1$, Eq. (10) takes the asymptotic form

$$\begin{aligned}\frac{E_g}{V} = \frac{2\pi\hbar^2}{m}n_0^2\tilde{a}\Big[\Big(1 + \kappa\frac{\tilde{b}}{\tilde{a}}\Big) + \frac{\tilde{a}}{\tilde{a}_{cr}} + \frac{128}{15}\left(\frac{n_0\tilde{a}^3}{\pi}\right)^{1/2}\frac{m^*}{m} \\ + 4\sqrt{\pi}\left(\frac{n_0\tilde{a}^3}{\pi}\right)^{1/2}\frac{m^*}{m}\tilde{R}\Big], \end{aligned} \tag{14}$$

with $m^* = \hbar^2/2td^2$ being the effective mass [28] associated with the band and $\tilde{a}_{cr} = -0.24d\sqrt{m/m^*}$ being a further renormalization of the scattering length due to the optical lattice. [6]. The first two terms in Eq. (14) are the mean-field contribution; whereas the remaining terms are generalized LHY correction in the presence of both optical lattice and disorder. In particular, the presence of disorder gives rise to the last term in Eq. (14). Compared to the free case, the periodic potentials affect the ground state energy in two ways. First, the periodic potential increases the local atomic density, thereby enhancing interatomic interactions characterized by the effective scattering length $\tilde{a} = C^2 a$. Second, the lattice modifies the dispersion relation, which is captured by the effective mass m^*. For vanishing disorder $\kappa = 0$, Eq. (14) is reduced to the corresponding result in Ref. [6]; whereas for vanishing optical lattice $s = 0$, our result recovers exactly the corresponding one in Ref. [26].

C. Quantum depletion

The average number of particles with nonzero momentum represents a depletion of the condensate which can be calculated by $N - N_0 = \sum_{\mathbf{k}\neq 0}\left(\epsilon^0_{\mathbf{k}} + \tilde{g}n_0 - E_{\mathbf{k}}\right)/2E_{\mathbf{k}}$. In the continuum limit, the quantum depletion of a BEC trapped in a 2D optical lattice in the presence of disorder is calculated as

$$\frac{N-N_0}{N} = \frac{1}{4\pi d^2}\sqrt{\frac{2m\tilde{g}}{n_0}}h\left(\frac{2t}{\tilde{g}n_0}\right) + \frac{\tilde{R}}{16\pi\hbar d^2}\sqrt{\frac{m\tilde{g}}{n_0}}K\left(\frac{2t}{\tilde{g}n_0}\right), \tag{15}$$

where $h(x)$ and $K(x)$ are functions of the variable $x = 2t/\tilde{g}n_0$ that are respectively defined as

$$h(x) = \int_{-\pi}^{\pi}\frac{d^2\mathbf{k}}{(2\pi)^2}\int_{x\gamma(\mathbf{k})}^{+\infty}\frac{dt}{\sqrt{t - x\gamma(\mathbf{k})}}\left[\frac{t+1}{\sqrt{t^2+2t}} - 1\right], \tag{16}$$

and

$$K(x) = \frac{\pi}{2}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\frac{{}_2F_1(\frac{1}{2},\frac{1}{2},1,\frac{x^2}{[1+x+x\sin^2(k/2)]^2})}{\left[1+x+x\sin^2(k/2)\right]\sqrt{x\sin^2(k/2)+1}}. \tag{17}$$

The first term in Eq. (15) arises from the two-body interactions of bosonic atoms trapped in the optical lattice. The second term, on the other hand, represents the effect of disorder on the quantum depletion.

In the 1D regime, $h(x) \simeq -\ln(2.7x)/\sqrt{2}$ so the first term in Eq. (15) diverges. It implies that in the absence of tunneling no real Bose-Einstein condensation exists which agrees with the general theorems in one dimension [30]. In the opposite 3D regime, the functions $h(x)$ and $K(x)$ respectively decay as $4/(3\pi\sqrt{2}x)$ and $2/(\pi x)$. Accordingly, the quantum depletion (15) takes the asymptotic form in the 3D regime:

$$\frac{N-N_0}{N} = \left(\frac{8}{3}+\frac{\tilde{R}\pi}{2}\right)\left(\frac{m^*}{m}\right)\left(\frac{n_0\tilde{a}^3}{\pi}\right)^{1/2}. \quad (18)$$

Eq. (18) generalizes the standard 3D result in the presence of weak disorder in free space to that in the presence of an optical lattice.

IV. REDUCTION OF SUPERFLUID DENSITY DUE TO COMBINED EFFECTS OF DISORDER AND OPTICAL LATTICE

A. Longitudinal and transverse response functions

In the second part of this paper, we apply the linear response theory to investigate the effects of disorder on the superfluid density of a BEC trapped in a 2D optical lattice. The general definition of the superfluid density is proposed by Hohenberg and Martin [34]. We emphasize that, unlike the condensate density, superfluidity is a kinetic property of a system and the superfluid density is not an equilibrium quantity but a transport coefficient, and should be determined by the response of momentum density to an externally imposed velocity field [34, 35]. According to Kubo's formula [36], the average momentum density $\langle g(\mathbf{r},t)\rangle$ induced by the external perturbation velocity field $\delta v(\mathbf{r}')$ is given by

$$\langle g_i(\mathbf{r},t)\rangle = \int_{-\infty}^{t} dt' \int d\mathbf{r}' R_{ij}(\mathbf{r}t,\mathbf{r}'t')\delta v_j(\mathbf{r}')e^{\epsilon t'}, \quad (19)$$

where

$$g(\mathbf{r},t) = \frac{\hbar}{2i}\left[\hat{\Psi}^\dagger(\mathbf{r})\nabla\hat{\Psi}(\mathbf{r}) - \hat{\Psi}(\mathbf{r})\nabla\hat{\Psi}^\dagger(\mathbf{r})\right], \quad (20)$$

is the momentum density operator, and

$$\begin{aligned} R_{ij}(\mathbf{r}t,\mathbf{r}'t') &= \left\langle \left[g_i(\mathbf{r},t), g_j(\mathbf{r}',t')\right]\right\rangle \\ &= \int \frac{d\mathbf{q}}{(2\pi)^3}\frac{d\omega}{2\pi} e^{i\left[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')\right]} R_{ij}(\mathbf{q},\omega), \end{aligned} \quad (21)$$

is the momentum density correlation function, averaged with the ground state wavefunction for a BEC at rest without perturbation at zero temperature. The static current-current response function is given by [37]

$$\begin{aligned} \chi_{ij}(\mathbf{q}) &= \frac{1}{m}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{R_{ij}(\mathbf{q},\omega)}{\omega - i\epsilon} \\ &= -2\sum_n \frac{\langle 0|J^i_{\mathbf{q}}|n\rangle\left\langle n|J^j_{-\mathbf{q}}|0\right\rangle}{\omega_{n0}}, \end{aligned} \quad (22)$$

where $\omega_{n0} = (\varepsilon_n - \varepsilon_0)/\hbar$ and $|0\rangle$ is the ground state and the sum is performed over a complete set of excited states $|n\rangle$ with the energy $\hbar\omega_n$. Here the current operator $J^i_{\mathbf{q}}$ is defined by

$$J^{x(y)}_{\mathbf{q}} = \frac{2td}{\hbar}\sum_{\mathbf{k}}\sin\left(k_{x(y)}+\frac{q_{x(y)}}{2}\right)\hat{a}^\dagger_{\mathbf{k}}a_{\mathbf{k}+\mathbf{q}}, \quad (23)$$

$$J^{z}_{\mathbf{q}} = \frac{\hbar}{2m}\sum_{\mathbf{k}}\left(k_z+\frac{q_z}{2}\right)\hat{a}^\dagger_{\mathbf{k}}a_{\mathbf{k}+\mathbf{q}}. \quad (24)$$

In general, the superfluid density of a BEC distinguishes between the low-frequency, long-wavelength longitudinal and transverse responses of the system [18, 30, 35, 36, 37]

$$\chi_{ij}(\mathbf{q}) = \frac{q_iq_j}{q^2}\chi_L(\mathbf{q}) + \left(\delta_{ij}-\frac{q_iq_j}{q^2}\right)\chi_T(\mathbf{q}). \quad (25)$$

The first term in Eq. (25) is the longitudinal component; it is parallel to $\mathbf{q}$ in both indices, i.e. $\sum_i q_i(q_iq_j/q^2) = q_j$. The second term, the transverse component is perpendicular to $\mathbf{q}$ in both indices, i.e. $\sum_i q_i(\delta_{ij} - q_iq_j/q^2) = 0$.

A longitudinal probe corresponds to boosting the system and the entire fluid responds. On the other hand, a low-frequency transverse probe corresponds to a slow rotation of the system, which only couples to the normal fluid while leaving the superfluid untouched. Hence the normal fluid density is defined in terms of the static transverse current-current correlation function as [18, 30, 35, 36, 37]

$$\rho_n = \lim_{q\to 0}\chi_T(\mathbf{q}). \quad (26)$$

The superfluid density is thereby defined as the difference between the total density ρ and the normal fluid density

$$\rho_s = \rho - \lim_{q\to 0}\chi_T(\mathbf{q}). \quad (27)$$

There is an important point to stress here. By using the particle conservation, one can obtain $\rho = \lim_{q\to 0}\chi_L(\mathbf{q})$ [18, 30, 35, 36, 37]. This result is fixed by the model-independent f-sum rule [30, 35, 36, 37]. One is therefore easily led to directly write $\rho_s = \lim_{q\to 0}[\chi_L(\mathbf{q}) - \chi_T(\mathbf{q})]$. This is not valid, however, for the calculation using the Bogoliubov approximation which violates the particle conservation by replacing $\hat{a}_0$ with a c-number $\sqrt{N_0}$ [18]. In particular, computations involving the time evolution of zero-momentum particle states would be false. The

non-zero momentum states, on the other hand, are not affected. Thus for our present calculation of the superfluid density within the Bogoliubov' theoretical framework, it is the calculation of Eq. (26) that can be trusted, not $\lim_{q\to 0}\chi_L(\mathbf{q})$. Accordingly, the superfluid density can only be determined through Eq. (27) in the present context.

B. Superfluid density

The transverse response of a BEC is only due to the normal fluid, since the superfluid component can only participate in irrotational flow. For a BEC trapped in a 2D optical lattice, the rotational symmetry is broken that gives rise to an anisotropic system. Consequently, the response in the unconfined z-direction is different from that in the confined x-y plane. Nevertheless, the system is still isotropic in the x-y plane. Hence it's more convenient to consider a slow rotation with respect to the z-axis. We therefore take the transverse response function along the z-direction, given by

$$\begin{aligned}\chi_{zz}(0) &= 2\sum_{\mathbf{k}}\frac{\hbar^2 k_z^2}{m}\frac{\varepsilon^0_{\mathbf{k}}}{E^4_{\mathbf{k}}}\langle|V_{\mathbf{k}}|^2\rangle \\ &= \frac{\tilde{R}}{4\hbar d^2}\sqrt{2m\tilde{g}n_0}\,I(\frac{2t}{\tilde{g}n_0}),\end{aligned} \qquad (28)$$

with $I(x)$ being a function of a variable $x = 2t/\tilde{g}n_0$ in the form

$$I(x)=\int_{-\pi}^{\pi}\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1}{\sqrt{x\gamma(\mathbf{k})+2}}\frac{1}{\left(\sqrt{x}+\sqrt{x\gamma(\mathbf{k})+2}\right)^2}. \qquad (29)$$

Two properties of the normal fluid density can immediately be stated based on Eq. (28): (i) the normal fluid component induced by disorder is not fixed by the long-wavelength properties of the excited quasi-particle. In order to assure the convergence of the integral in Eq. (28), the behavior of the elementary excitation spectrum at high momenta is important; (ii) Eq. (28) can be interpreted as the second-order term in the perturbation expansion of the normal fluid density in terms of weak disorder $V_{\mathbf{k}}$. These two properties should be general and do not depend on the concrete form of disorder [18].

The superfluid density is thereby found to be $\rho_s = \rho - \rho_n$ with $\rho_n = \chi_{zz}(0)$. In the region $x \gg 1$, $I(x)$ asymptotically approaches the function $\sqrt{2}/(6\pi x)$. Eq. (28) therefore takes the asymptotic form

$$\chi_{zz}(0) = \frac{m^*}{m}\frac{2\sqrt{\pi}}{3}\tilde{R}n_0\left(n_0\tilde{a}^3\right)^{1/2}. \qquad (30)$$

Eq. (30) generalizes the normal fluid density induced by disorder in the free space to that in the presence of optical lattice. Accordingly, we conclude that in the asymptotic anisotropic 3D regime, the disorder still generates an amount of normal fluid that is equal to 4/3 of the condensate depletion [18], despite the presence of optical lattice.

To conclude this section, we note that in using Eq. (27) to determine the superfluid density, one has to specify detailed probing approach and the limiting procedures thereof to distinguish transverse and longitudinal responses [18, 30, 37]. In rotationally invariant and uniform systems, definition (27) has been a standard expression to calculate the superfluid density [18, 30]. It would be less advantageous, however, for nonuniform systems. There are other definitions for superfluid density [3, 38, 39]. Despite the formal difference, these various definitions are fundamentally grounded in analyzing the linear response of a fluid to an external velocity boost. The underlying key quantity to determine superfluid components, therefore, is the momentum density correlation function. In isotropic and uniform systems, these definitions all lead to the same superfluid fraction within the linear response approximation. To determine the superfluid density in nonuniform systems, Refs. [3, 38, 39] gave a more general expression. In the present context, both the definition (27) and the definition in Refs. [3, 38, 39] are appropriate. This is because the presence of optical lattice has been accounted for through renormalizing the scattering length $\tilde{g}$ and the kinetic energy term $\varepsilon^0_{\mathbf{k}}$ [28]. As a result, despite the presence of lattice, one can study the quantum behaviors of a BEC as if the space is homogeneous [28, 29]. In particular, in the limit $8t \gg \mu$ the effect of lattice is explicitly captured by the effective mass m^* and the effective interaction $\tilde{g}$. Consequently in the asymptotic 3D region, a BEC trapped in an optical lattice can be effectively regarded as a uniform fluid composed of bosons with an effective mass m^* and an effective coupling constant $\tilde{g}$ [6, 28]. Thus within our present context, Eq. (27) still gives a good definition for the superfluid density particularly in the asymptotic 3D region, despite the presence of lattice.

V. POSSIBLE EXPERIMENTAL SCENARIOS AND CONCLUSION

In this work, the physics of our model is captured by the interplay among three quantities: the strength of an optical lattice s, the interaction between bosonic atoms $\tilde{g}n_0$, and the strength of disorder $\tilde{R}$. All these quantities are experimentally controllable using state-of-the-art technologies. The interatomic interaction can be controlled in a very versatile manner via the technology of Feshbach resonances. In the typical experiments to date, the values of ratio $\tilde{g}n_0/E_R$ range from 0.02 to 1 [1, 2]. The depth of an optical lattice s can be changed from $0E_R$ to $32E_R$ almost at will [4]. Disorder may be created in a repeatable way by introducing impurities in the sample [21], or using laser speckles and multi-chromatic lattices [19, 20, 23, 24]. Therefore, the phenomena discussed in this paper should be observable within the current ex-

perimental capability.

We emphasize here that our investigation of a BEC in the presence of both optical lattice and weak disorder has been done within the Bogoliubov theory. Further improvement of the theoretical framework is to permit the treatment of the system properties for whole range of interatomic interaction strength, from zero to infinity, as well as arbitrarily strong disorder [7].

In summary, a BEC trapped in a 2D optical lattice in the presence of disorder is analytically investigated within the framework of the Bogoliubov theory. We focus on analyzing the combined effects of disorder and optical lattice on the quantum fluctuations and superfluid density of the BEC system. Accordingly, the analytical expressions of the ground state energy and the quantum depletion of the system are obtained. Our results show that the 2D lattice induces a characteristic 3D to 1D crossover in the behavior of quantum fluctuations. Furthermore, the effects of disorder on this dimensional crossover are discussed in detail. Especially, we argue that our results in the asymptotic 1D regime is a generalized Lieb-Linger solution of the 1D model expanded in the weak coupling regime in the presence of weak disorder. In addition, the normal fluid density of the condensate induced by the disorder is obtained by calculating the transverse response function. Our results show that in the 3D regime, the quantum depletion due to the combined effects of disorder and lattice will asymptotically approach $3/4$ of the corresponding reduction of the superfluid density. Finally, the conditions for possible experimental realization of our scenario are discussed.

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